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On the number of countably compact group topologies on a free Abelian group

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Abstract

We show under MA(σ-centered) the existence of at least $(2^\omega)^+$ non-homeomorphic topological group topologies on the free Abelian group of size 2^ω which make it countably compact and separable. In particular, under GCH the maximum possible number of such topologies is attained. As a corollary, we show the existence of a semigroup which possesses $(2^\omega)^+$ non-homeomorphic semigroup topologies which make it a counterexample for Wallace's Problem. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

It is well known that a free Abelian group cannot be endowed with a compact group topology (see [9]). Many authors (Dikranjan and Shakmatov [2], Tkačenko [9], Comfort and Remus [1]) have shown the existence of pseudocompact group topologies on a free Abelian group. Tkačenko [9] showed under the Continuum Hypothesis, the existence of a countably compact topological group topology on the free Abelian group of size $\mathfrak{c}$. We showed in [13] that the existence of an initially ω_1-compact group topology on the free Abelian group of size $\mathfrak{c}$ is independent of ZFC.

In this paper we show under MA(σ-centered) that the free Abelian group can be endowed with $\mathfrak{c}^+$ non-homeomorphic group topologies which make it countably compact.

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We will denote by $\mathbb{T}$ the unitary circle group as a subspace of $\mathbb{R}^2$. We use the additive notation and the identity will be denoted by 0.

2. A non-homeomorphic topology for the free Abelian group

2.1. Producing a non-homeomorphic topology

The main result in this work is the following.

Theorem 2.1. (MA(σ-centered)) *Let $\{(X_\alpha, \mathcal{T}_\alpha)\colon \alpha < \mathfrak{c}\}$ be a family of topological spaces of size $\mathfrak{c}$. Then the free Abelian group G generated by $\mathfrak{c}$ generators can be endowed with a separable countably compact topological group topology $\mathcal{T}$ such that $(G, \mathcal{T})$ is not homeomorphic to $(X_\alpha, \mathcal{T}_\alpha)$ for any $\alpha < \mathfrak{c}$.*

From Theorem 2.1, we have the following:

Example 2.2. (MA(σ-centered)) There exist at least $\mathfrak{c}^+$ non-homeomorphic topological group topologies on the free Abelian group of size $\mathfrak{c}$ which make it separable and countably compact.

Definition 2.3. Let X be a topological space and for each $x \in X$ let $\mathcal{N}_x$ be a basis for x. If $x \in X$ is an accumulation point of $\{x_n\colon n \in \omega\} \subseteq X$ then we denote by

$$\mathcal{F}\big(X, \{x_n\colon n \in \omega\}, x\big) = \big\{A \subseteq \omega\colon (\exists U \in \mathcal{N}_x)\{n \in \omega\colon x_n \in U\} \subseteq A\big\}.$$

Let $\mathcal{F}(X)$ be the family of all filters $\mathcal{F}(X, \{x_n\colon n \in \omega\}, x)$, where $x \in X$ is an accumulation point of the sequence $\{x_n\colon n \in \omega\} \subseteq X$.

Lemma 2.4. *Let $h : X \to Y$ be a homeomorphism of the space X onto Y. If $x \in X$ is an accumulation point of a sequence $\{x_n\colon n \in \omega\} \subseteq X$ then $h(x)$ is an accumulation point of the sequence $\{h(x_n)\colon n \in \omega\} \subseteq Y$ and*

$$\mathcal{F}\big(X, \{x_n\colon n \in \omega\}, x\big) = \mathcal{F}\big(Y, \{h(x_n)\colon n \in \omega\}, h(x)\big).$$

Note that if X and Y are homeomorphic spaces then $\mathcal{F}(X \times X) = \mathcal{F}(Y \times Y)$. This fact will be the key to produce a non-homeomorphic topology.

Fix a family $\{(X_\alpha, \mathcal{T}_\alpha)\colon \alpha < \mathfrak{c}\}$ as in Theorem 2.1. We will construct a subset $Z = \{z_\alpha\colon \alpha < \mathfrak{c}\} \subseteq \mathbb{T}^{\mathfrak{c}}$ such that the subgroup G generated by Z endowed with the subspace topology $\mathcal{T}$ will satisfy the conclusion of Theorem 2.1.

The construction of Z is by induction and at stage $\xi \geqslant \omega$, we will have defined $\{z_\alpha\restriction_\xi\colon \alpha < \xi\}$.

If Z is such that $(0, 0)$ is an accumulation point of $\{(z_{2n}, z_{2n+1})\colon n \in \omega\}$ and

$$\mathcal{F} = \mathcal{F}\big((\mathbb{T}^{\mathfrak{c}})^2, \big\{(z_{2n}, z_{2n+1})\colon n \in \omega\big\}, (0, 0)\big) \notin \mathcal{F}(X_\beta \times X_\beta), \quad \text{for any } \beta < \mathfrak{c} \qquad (*)$$

then G is not homeomorphic to X_β for any $\beta < \mathfrak{c}$.

Enumerate $\bigcup_{\beta<\mathfrak{c}} \mathcal{F}(X_\beta \times X_\beta)$ as $\{\mathcal{F}(\xi)\colon \omega \leqslant \xi < \mathfrak{c}\}$. At stage $\xi+1$, we want to make sure that $\mathcal{F} \neq \mathcal{F}(\xi)$.

Note that at stage ξ, the filter $\mathcal{F}$ is not completely determined. In fact, $\mathcal{F}\restriction_\xi = \mathcal{F}((\mathbb{T}^\xi)^2, \{(z_{2n}\restriction_\xi, z_{2n+1}\restriction_\xi)\colon n \in \omega\}, (0,0))$ is the set of all elements of $\mathcal{F}$ which are determined at stage ξ and since $\mathfrak{c}$ is a limit ordinal, $\mathcal{F} = \bigcup_{\xi<\mathfrak{c}} \mathcal{F}\restriction_\xi$.

Thus, $(G, \mathcal{T})$ satisfies $(*)$ if

$$\mathcal{F}\restriction_{\xi+1} \not\subseteq \mathcal{F}(\xi) \tag{$**$}$$

holds for every $\xi \in [\omega, \mathfrak{c})$.

2.2. The construction

We need to fix some enumerations to proceed.

Definition 2.5. Let $\{g_\alpha\colon \omega \leqslant \alpha < \mathfrak{c}\}$ be an enumeration of the set of all functions from non-empty finite subsets of $\mathfrak{c}$ into $\mathbb{Z} \setminus \{0\}$ such that $\mathit{dom}\, g_\alpha \subseteq \alpha$.

Definition 2.6. Let $\mathcal{G}$ be the family of all sequences $\{f_n\colon n \in \omega\}$ such that f_n is a function from a non-empty subset of $\mathfrak{c}$ into $\mathbb{Z} \setminus \{0\}$ for each $n \in \omega$ and
either

(i) there exists a finite subset F of $\mathfrak{c}$ such that $\mathit{dom}\, f_n = F$, for every $n \in \omega$, or
(ii) $\mathit{dom}\, f_n \neq \mathit{dom}\, f_m$, for each $n, m \in \omega$

are satisfied.

Definition 2.7. Let $\{\mathcal{G}_\alpha\colon \omega \leqslant \alpha < \mathfrak{c}\}$ be an enumeration of $\mathcal{G}$ in such a way that $\bigcup_{f \in \mathcal{G}_\alpha} \mathit{dom}\, f \subseteq \alpha$.

We are ready now to state the inductive hypothesis for the construction of the subset $Z \subseteq \mathbb{T}^\mathfrak{c}$.

Lemma 2.8. *Let $Z = \{z_\alpha\colon \alpha < \mathfrak{c}\} \subseteq \mathbb{T}^\mathfrak{c}$ be a family such that the following are satisfied*:

(a) $\sum_{\xi \in \mathit{dom}\, g_\beta} g_\beta(\xi) z_\xi(\beta) \neq 0$ *for each* $\beta \in [\omega, \mathfrak{c})$.
(b) $z_\alpha\restriction_\beta$ *is an accumulation point of* $\{\sum_{\xi \in \mathit{dom}\, f} f(\xi) z_\xi\restriction_\beta\colon f \in \mathcal{G}_\alpha\}$ *for each* $\beta \in \mathfrak{c}$ *and* $\alpha < \beta$.
(c) $\{z_n\restriction_\beta\colon n \in \omega\}$ *is dense in* $\mathbb{T}^\beta$, *for each* $\beta < \mathfrak{c}$.
(d) *There exists* $A_\beta \in [\omega]^\omega$ *for each* $\beta \in [\omega, \mathfrak{c})$ *such that*
- $A_\beta \notin \mathcal{F}(\beta)$;
- $(0,0)$ *is an accumulation point for the sequence*
$$\{(z_{2n}\restriction_\beta, z_{2n+1}\restriction_\beta)\colon n \in A_\beta\};$$
- *there exists a neighbourhood* U *of* $0 \in \mathbb{T}$ *such that*
$$\{n \in \omega \setminus A_\beta\colon (z_{2n}\restriction_\beta, z_{2n+1}\restriction_\beta) \in U \times U\}$$
is finite.

Then the group G generated by $Z \subseteq \mathbb{T}^\mathfrak{c}$ satisfies the conclusion of Theorem 2.1.

Proof. Since $\{z_\alpha\colon \alpha < \mathfrak{c}\}$ satisfies conditions (a)–(c), the group G is free Abelian, and as a subspace of $\mathbb{T}^{\mathfrak{c}}$, it is countably compact and separable (see [13]).

Condition (d) implies that condition $(**)$ is satisfied. Thus G is not homeomorphic to $(X_\alpha, \mathcal{T}_\alpha)$ for any $\alpha < \mathfrak{c}$. □

Lemma 2.9. (MA(σ-centered)) *There exists* $\{z_\alpha\colon \alpha < \mathfrak{c}\} \subseteq \mathbb{T}^{\mathfrak{c}}$ *satisfying the conditions of Lemma* 2.8.

Proof. The functions are constructed by induction.

At stage ω, fix a dense subset $\{z_n\restriction_\omega\colon n \in \omega\} \subseteq \mathbb{T}^\omega$. Clearly (a)–(d) are satisfied.

Suppose that we have constructed $\{z_\alpha\restriction_\gamma\colon \alpha < \gamma\}$ for each $\gamma < \beta$.

Suppose first that β is limit. Define $z_\alpha\restriction_\beta = \bigcup_{\alpha<\gamma<\beta} z_\alpha\restriction_\gamma$. Then $\{z_\alpha\restriction_\beta\colon \alpha < \beta\}$ satisfies conditions (a)–(d).

Suppose that $\beta = \gamma + 1$. Fix an accumulation point $z_\gamma\restriction_\gamma \in \mathbb{T}^\gamma$ of the sequence

$$\left\{\sum_{\xi \in dom\, f} f(\xi) z_\xi\restriction_\gamma\colon f \in \mathcal{G}_\gamma\right\} \subseteq \mathbb{T}^\gamma.$$

We will construct a function $\phi\colon\beta \to \mathbb{T}$ and define $z_\xi(\gamma) = \phi(\xi)$. Let us first discuss condition (d).

Every $A \in \mathcal{F}\restriction_\gamma$ is infinite and $\mathcal{F}\restriction_\gamma$ has a filter basis of size $< \mathfrak{c}$. Applying MA(σ-centered), there exists an infinite subset A_γ of ω such that $A_\gamma \subseteq^* A$ for each $A \in \mathcal{F}\restriction_\gamma$. Then $(z_{2n}\restriction_\gamma, z_{2n+1}\restriction_\gamma)\colon n \in A_\gamma\}$ has $(0,0)$ as an accumulation point. We can assume that $A_\gamma \notin \mathcal{F}(\gamma)$.

It suffices to define $\phi(n) = z_n(\gamma)$ for each $n \in \omega$ so that

(A) For each $k \in \omega$, for each $m \in \omega$ there exists $n \in A_\gamma \setminus m$ such that $\|\phi(2n)\| < 1/(k+1)$ and $\|\phi(2n+1)\| < 1/(k+1)$.

(B) There exists an open neighbourhood of O of the identity in $\mathbb{T}$ such that the set $\{n \in \omega \setminus A_\gamma\colon \phi(2n) \in O \wedge \phi(2n+1) \in O\}$ is finite.

The conditions on ϕ to have (a)–(c) satisfied are as in [13]. We have to recall some definitions:

Definition 2.10. For each $\alpha < \beta$, $F \in [\gamma]^{<\omega}$ and $k \in \omega$, let

$$\{E(\alpha, F, k, m)\colon m \in \omega\}$$

be a partition of

$$\left\{f \in \mathcal{G}_\alpha\colon \forall \mu \in F\left(\left\|\sum_{\xi \in dom\, f} f(\xi) z_\xi(\mu) - z_\alpha(\mu)\right\| < \frac{1}{k+1}\right)\right\}$$

into subsets of infinite size.

Fix a basis $\mathcal{B}_\gamma$ for $\mathbb{T}^\gamma$ of size $< \mathfrak{c}$ and a countable basis $\mathcal{B}$ for $\mathbb{T}$ whose elements are not empty.

Definition 2.11. For each $U \in B_\gamma$ let $\{D(U,k)\colon k \in \omega\}$ be a partition of $\{n \in \omega\colon z_n \restriction_\gamma \in U\}$ into subsets of infinite size.

To have (a) satisfied, it suffices that
(C) $\sum_{\xi \in dom\, g_\gamma} g_\gamma(\xi)\phi(\xi) \neq 0$.
To have (b) satisfied, it suffices that
(D) For each $\alpha < \beta$, $F \in [\gamma]^{<\omega}$ and $k, m \in \omega$, the set

$$\left\{ f \in E(\alpha, F, k, m)\colon \left\| \sum_{\xi \in dom\, f} f(\xi)\phi(\xi) - \phi(\alpha) \right\| < \frac{1}{k+1} \right\}$$

is not empty.
To have (c) satisfied, it suffices that
(E) $\{n \in D(U,k)\colon \phi(n) \in W\}$ is not empty, for each $W \in \mathcal{B}$.
Thus the proof of Lemma 2.9 will be finished if we show the following:

Lemma 2.12. (MA(σ-centered)) *There exists a function $\phi : \beta \to \mathbb{T}$ such that conditions* (A)–(E) *are satisfied.*

We will prove Lemma 2.12 in the next subsection. □

2.3. The successor stage

We will use a σ-centered partial order $\mathbb{P}$ and dense subsets of $\mathbb{P}$ to construct ϕ.

Definition 2.13. Given $\mathcal{B}' \in [\mathcal{B}]^{<\omega} \setminus \emptyset$, define

$$\sum \mathcal{B}' = \left\{ \sum_{U \in \mathcal{B}'} x_U\colon \{x_U\colon U \in F\} \in \prod \mathcal{B}' \right\}.$$

Definition 2.14. Let r be a function from a non-empty finite subset of β into $\mathcal{B}$ such that $dom\, g_\gamma \subseteq dom\, r$ and $0 \notin \sum_{\xi \in dom\, g_\gamma} g_\gamma(\xi) r(\xi)$. Let

$$O = \left\{ x \in \mathbb{T}\colon \|x\| < \tfrac{1}{8} \right\}.$$

Definition 2.15. Let $\mathbb{P}$ be the set of all functions $p : F \to \mathcal{B}$ such that F is a finite subset of β, $dom\, p \cap \omega = 2n$ for some $n \in \omega$.

Denote by K_p the unique integer such that $dom\, p \cap \omega = 2K_p$.

Given $p, q \in \mathbb{P}$, we define $p \leqslant q$ if and only if $dom\, p \supseteq dom\, q$, $\forall \xi \in dom\, q$, either $p(\xi) \subseteq \overline{q(\xi)}$ or $p(\xi) = q(\xi)$ and $\forall n \in [K_q, K_p) \setminus A_\gamma$ there exists $i \in 2$ such that $p(2n+i) \cap O = \emptyset$.

Lemma 2.16. *The partial order $\mathbb{P}$ is σ-centered.*

Proof. Consider $\mathcal{B}$ as a discrete space and let A be a dense subset of $\mathcal{B}^\beta$. Let $\{p_n\colon n \in \omega\}$ an enumeration of all finite functions from $2n$ into $\mathcal{B}$, where $n \in \omega$. For each $s \in A$ and

$n \in \omega$, let $\mathbb{P}_{s,n} = \{p \in \mathbb{P}\colon\ p \subseteq s \wedge p\restriction_\omega = p_n\}$. Then $\{\mathbb{P}_{s,n}\colon\ s \in A \wedge n \in \omega\}$ witnesses that $\mathbb{P}$ is σ-centered. $\square$

Definition 2.17. For each $\xi < \beta$, $\mathcal{D}_\xi = \{p \in \mathbb{P}\colon\ \xi \in dom\, p\}$.

Definition 2.18. For each $\alpha < \beta$, $F \in [\gamma]^{<\omega}$ and $k, m \in \omega$, define

$$\mathcal{E}(\alpha, F, k, m) = \Bigg\{ p \in \mathbb{P}\colon\ \exists f \in E(\alpha, F, k, m) \bigg(dom\, f \subseteq dom\, p \\ \text{such that } \bigg\| \sum_{\xi \in dom\, f} f(\xi) p(\xi) - p(\alpha) \bigg\| < \frac{1}{k+1} \bigg) \Bigg\}.$$

Definition 2.19. For each $U \in \mathcal{B}_\gamma$, $k \in \omega$ and $W \in \mathcal{B}$, define $\mathcal{D}(U, k, W) = \{p \in \mathbb{P}\colon\ \exists n \in D(U, k) \cap dom\, p$ such that $p(n) \subseteq W\}$.

Definition 2.20. For each $k, m \in \omega$ define

$$\mathcal{I}_m = \Bigg\{ p \in \mathbb{P}\colon\ \exists n \in (K_p \cap A_\gamma) \setminus m \\ \text{such that } \big\| p(2n) \big\| < \frac{1}{k+1} \wedge \big\| p(2n+1) \big\| < \frac{1}{k+1} \Bigg\}.$$

We leave for the reader to check that the subsets above defined are dense.

Lemma 2.21. *The subsets of* $\mathbb{P}$ *in Definitions* 2.17–2.20 *are dense in* $\mathbb{P}$.

Proof of Lemma 2.12. Let $\mathbb{P}$ be the partial order in Definition 2.15. We have fewer than $\mathfrak{c}$ dense subsets in Definitions 2.17–2.20. By Lemma 2.16 and MA(σ-centered), there exists a generic filter $\mathbb{G}$ which intersects each dense set in Definitions 2.17–2.20.

For each $\xi < \beta$, $\mathbb{G} \cap D_\xi \neq \emptyset$. Hence, the set $\{p \in \mathbb{G}\colon\ \xi \in dom\, p\}$ is not empty. By the ordering of $\mathbb{P}$ and the fact that $\mathbb{G}$ is a filter, one can show that

$$\Phi(\xi) = \bigcap \big\{ p(\xi)\colon\ \xi \in dom\, p \wedge p \in \mathbb{G} \big\}$$

is not empty. Thus, we can define $\phi : \beta \to \mathbb{T}$ such that $\phi(\xi) \in \Phi(\xi)$ for each $\xi < \beta$.

Let us check now that condition (A)–(E) are satisfied by ϕ. Condition (A) (respectively (D), (E)) is satisfied, since $\mathbb{G}$ intersects each dense subset in Definition 2.20 (respectively Definitions 2.18, 2.19).

Note that

$$\sum_{\xi \in dom\, g_\gamma} g_\gamma(\xi) \phi(\xi) \in \sum_{\xi\ dom\, g_\gamma} g_\gamma(\xi) r(\xi) \not\ni 0.$$

Thus, condition (C) is satisfied.

Fix $q \in \mathbb{G}$. Let $n \in (\omega \setminus A_\gamma)$ be greater or equal to K_q. Then there exists $p \in \mathbb{G}$ such that $K_p > n$ and $p \leqslant q$. Then there exists $i \in 2$ such that $\phi(2n+i) \in p(2n+i) \subseteq \mathbb{T} \setminus O$. Thus condition (B) is satisfied. $\square$

3. Non-homeomorphic semigroup topologies and the Wallace problem

Definition 3.1 (*after Robbie and Svetlichny* [8]). A topological semigroup is a Wallace semigroup if it is
(1) cancellative,
(2) countably compact,
(3) not a topological group.

Wallace [16] asked whether or not there existed such a semigroup. The motivation came from the well-known fact that every topological semigroup which is cancellative and compact must be a topological group. He pointed out that there were some printed alleged proofs of the non-existence of Wallace semigroups but those were not correct.

Robbie and Svetlichny [7] were the first to show the existence of a Wallace semigroup, under the Continuum Hypothesis. In [10], we constructed a Wallace semigroup using the following statement: *The real line is not the union of fewer than* $\mathfrak{c}$ *nowhere dense subsets* which is equivalent to Martin's Axiom for countable partial orders ($\mathrm{MA}_{\text{countable}}$).

Example 3.2. ($\mathrm{MA}(\sigma$-centered)) There exist a semigroup and $\mathfrak{c}^+$ semigroup topologies which make it a Wallace semigroup.

Proof. Let G be the free Abelian group of size $\mathfrak{c}$ and let $\{s_\alpha\colon \alpha < \mathfrak{c}\}$ be a set of generators which witnesses that G is free Abelian. Let $\mathcal{H}$ be the set of all functions $g : F \to \mathbb{N}$ with $F \in [\mathfrak{c}]^{<\omega}$ and let

$$S = \left\{ \sum_{\xi \in dom\, g} g(\xi) s_\xi \colon\ g \in \mathcal{H} \right\}.$$

Clearly S is cancellative and it is not a group.

Let $\{\mathcal{T}_\alpha\colon \alpha < \mathfrak{c}\}$ be a family of semigroup topologies on S. Let Z be the subset from the Lemma 2.8. Then the semigroup $Y = \{\sum_{\xi \in dom\, g} g(\xi) z_\xi\colon\ g \in \mathcal{H}\}$ is algebraically the same as S. From the proof of Lemma 2.8, $Y \subseteq \mathbb{T}^{\mathfrak{c}}$ as a subspace, is also countably compact and it is not homeomorphic to $(S, \mathcal{T}_\alpha)$ for any $\alpha < \mathfrak{c}$.

Therefore, there are at least $\mathfrak{c}^+$ non-homeomorphic semigroup topologies on S which make it countably compact (thus a Wallace semigroup). □

4. Final remarks

We have shown in our thesis [15] (see [14]) the existence, under $\mathrm{MA}_{\text{countable}}$, of a Wallace semigroup whose square is not countably compact. Unaware of our result, Robbie and Svetlichny [8] showed at the TOPOSYM 96 by a different method, the existence under CH, of two Wallace semigroups whose product is not countably compact.

Also in [14], we showed the existence, under $\mathrm{MA}(\sigma$-centered), of a group topology on the free Abelian group G of size $\mathfrak{c}$ which make G countably compact but G^2 not countably compact.

We could modify the construction from Section 2 to obtain the following:

Example 4.1. (MA(σ-centered)) There exist at least $\mathfrak{c}^+$ non-homeomorphic topological group topologies on the free Abelian group of size $\mathfrak{c}$ which make it separable, countably compact and its square not countably compact.

Example 4.2. (MA(σ-centered)) There exist a semigroup and $\mathfrak{c}^+$ semigroup topologies which make it a Wallace semigroup whose square is not a Wallace semigroup.

In [3], van Douwen constructed under MA, a subgroup G of $2^{\mathfrak{c}}$ of cardinality $\mathfrak{c}$ such that G is separable, countably compact and for which every convergent sequence is trivial. He also showed that G contains two countably compact subgroups whose product is not countably compact. Note that G is a vector space over the field 2.

We can use the method presented in Section 2 to show that under MA(σ-centered), the vector space over 2 of cardinality $\mathfrak{c}$ has at least $\mathfrak{c}^+$ group topologies which make it separable, countably compact, every convergent sequence is trivial and whose square is not countably compact. The method in Section 2 could also be applied to obtain a group and non-homeomorphic group topologies with the properties of the examples in [11,12].

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